Sp(n), THE SYMPLECTIC GROUP

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1. Introduction

1.1. Definition. $Sp(n) = \mathcal{O}(n, \mathbb{H}) = \{A \in M_n(\mathbb{H}) | A^{\mathrm{t}}\bar{A} = I\}$ is the symplectic group.

1.2. Example.

$$\begin{aligned} Sp(1) &= \{z \in M_1(\mathbb{H}) || z| = 1\} \\ &= \{z = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d | a^2 + b^2 + c^2 + d^2 = 1\} \\ &\cong S^3 \end{aligned}$$

2. The Lie Algebra sp(n)

2.1. Definition. A matrix $A \in M_n(\mathbb{H})$ is skew - symplectic if $\mathbf{A} + {}^{\mathrm{t}}\bar{\mathbf{A}} = 0$.

2.2. Definition. $sp(n) = \{A \in M_n(\mathbb{H}) | A + {}^{\mathrm{t}}\bar{A} = 0\}$ is the Lie algebra of Sp(n) with commutator bracket [A,B] = AB - BA.

Proof. This was proven in class. □

2.3. Fact. sp(n) is a real vector space.

Proof. Let $A, B \in sp(n)$ and $a, b \in \mathbb{R}$
$(aA + bB) + {}^{\mathrm{t}}\overline{aA + bB} = a(A + {}^{\mathrm{t}}\bar{A}) + b(B + {}^{\mathrm{t}}\bar{B}) = 0$ □

2.4. **Fact.** The dimension of $sp(n)$ is $(2n+1)n$.

Proof. Let $A \in sp(n)$.

$$\mathbf{A} = \begin{pmatrix} a_{11} + \mathbf{i}b_{11} + \mathbf{j}c_{11} + \mathbf{k}d_{11} & a_{12} + \mathbf{i}b_{12} + \mathbf{j}c_{12} + \mathbf{k}d_{12} & \dots & a_{1n} + \mathbf{i}b_{1n} + \mathbf{j}c_{1n} + \mathbf{k}d_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + \mathbf{i}b_{n1} + \mathbf{j}c_{n1} + \mathbf{k}d_{n1} & a_{n2} + \mathbf{i}b_{n2} + \mathbf{j}c_{n2} + \mathbf{k}d_{n2} & \dots & a_{nn} + \mathbf{i}b_{nn} + \mathbf{j}c_{nn} + \mathbf{k}d_{nn} \end{pmatrix}$$

Then $\mathbf{A} + {}^{\mathrm{t}}\bar{\mathbf{A}} =$

$$\begin{pmatrix} 2a_{11} & (a_{12}+a_{21}) + \mathbf{i}(b_{12}-b_{21}) + \mathbf{j}(c_{12}-c_{21}) + \mathbf{k}(d_{12}-d_{21}) & \dots & (a_{1n}+a_{n1}) + \mathbf{i}(b_{1n}-b_{n1}) + \mathbf{j}(c_{1n}-c_{n1}) + \mathbf{k}(d_{1n}-d_{n1}) \\ (a_{21}+a_{12}) + \mathbf{i}(b_{21}-b_{12}) + \mathbf{j}(c_{21}-c_{12}) + \mathbf{k}(d_{21}-d_{12}) & 2a_{22} & \dots & (a_{2n}+a_{n2}) + \mathbf{i}(b_{2n}-b_{n2}) + \mathbf{j}(c_{2n}-c_{n2}) + \mathbf{k}(d_{2n}-d_{n2}) \\ \vdots & & \ddots & \vdots \\ (a_{1n}+a_{n1}) + \mathbf{i}(b_{1n}-b_{n1}) + \mathbf{j}(c_{1n}-c_{n1}) + \mathbf{k}(d_{1n}-d_{n1}) & \dots & & 2a_{nn} \end{pmatrix}$$

$A + {}^{\mathrm{t}}\bar{A} = 0$, so:

$a_{xx} = 0, \forall x \rightarrow 0$ degrees of freedom
$a_{xy} = -a_{yx}, x \neq y \rightarrow \frac{n(n-1)}{2}$ degrees of freedom
$b_{xy} = b_{yx}, x \neq y \rightarrow \frac{n(n-1)}{2}$ degrees of freedom
$c_{xy} = c_{yx}, x \neq y \rightarrow \frac{n(n-1)}{2}$ degrees of freedom
$d_{xy} = d_{yx}, x \neq y \rightarrow \frac{n(n-1)}{2}$ degrees of freedom
b_{xx}, c_{xx}, d_{xx} unrestricted, $\forall x \rightarrow$ 3n degrees of freedom

In total, $dim(sp(n)) = 2n^2 + n = n(2n+1)$ □

2.5. **Proposition.** If $A \in sp(n)$, then $e^A \in Sp(n)$, ie. $exp : sp(n) \to Sp(n)$

Proof. $I = e^0 = e^{A+{}^t\bar{A}} = e^A e^{{}^t\bar{A}} = e^A \cdot {}^t\overline{e^A}$

□

2.6. **Proposition.** If $G = Sp(n)$, then $T_G = sp(n)$

Proof. The proof was done in class.
$T_G \subset${skew matrices} and {skew matrices} $\subset T_G$, so $T_G =$ {skew matrices}.

□

2.7. **Corollary.** $dim(Sp(n)) = (2n+1)n$

Proof. $dim(Sp(n)) = dim(T_{Sp}) = dim(sp(n)) = (2n+1)n$

□

3. Interesting Isomorphisms

3.1. **Invariants.** In order for a group to be isomorphic to Sp(n), it must have the same invariants to preserve structure. Rank and dimension are numerical invariants. The center is a subgroup invariant.

When comparing Sp(n) to other matrix groups with the same rank for some rank ≥ 4, dim U $<$ dim SU $<$ dim SO(even) $<$ dim SO(odd) $=$ dim Sp [1]

Thus, for odd dimensions, $SO(n) = \{A \in O(n) | \text{det A} = 1\}$ might be isomorphic. However, SO(n) has center {I}, while Sp(n) has center ± 1, so that means for rank ≥ 4, Sp(n) is not isomorphic to U(n), SU(n), and SO(n).

What about for rank 1, 2, and 3?
For rank 1: Sp(1), SU(2), and SO(3) all have dimension 3. Sp(1) and SO(3) are not isomorphic because they have different centers, but as proved in the homework [PS 2], SO(3) $\cong Sp(1)/\{\pm I\}$.

3.2. **Proposition.** Sp(1) $\cong SU(2) = \{A \in U(2) | \text{det A} = 1\}$

Proof. Let z $\in$ Sp(1), $z = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$.
In our homework [PS 1] we showed the map $\phi : \mathbb{H} \to M_2(\mathbb{C})$ given by
$\phi(a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d) = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$
is an injective algebra homomorphism.

To show that the map is also an isomorphism $\phi : Sp(1) \to$ SU(2), I show
1) $\forall z, z \in$ Sp(1), $\phi(z) \in SU(2)$
Let $z = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$. Then $a^2 + b^2 + c^2 + d^2 = 1$.

$$
\begin{aligned}
\phi(z)^{\mathrm{t}}\overline{\phi(z)} &= \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \begin{pmatrix} a-ib & -c-id \\ c-id & a+ib \end{pmatrix} \\
&= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

$\det(\phi(z)) = 1$. Thus, $\phi(z) \in SU(2)$.

2) Next I show ϕ is surjective, ie. for every A $\in$ SU(2), there is some z $\in$ Sp(1) such that $A = \phi(z)$.
Let A $= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU(2), \alpha, \delta, \beta, \gamma \in \mathbb{C}$
det(A)=1, so $\alpha\delta - \beta\gamma = 1$.
Also, $A \in$ SU(2) $\subset U(2) = \mathcal{O}(2, \mathbb{C})$, so the rows of A form an orthonormal basis for $\mathbb{C}^2$.
Thus, $\delta = \bar{\alpha}$ and $\gamma = -\bar{\beta}$.
Let $\alpha = a + ib$ and $\beta = c + id$, then set $z = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$.
z $\in$ Sp(1) since $a^2 + b^2 + c^2 + d^2 = 1$. Also, $A = \phi(z)$.

□

For rank 2, Sp(2) and SO(5) both have dimension 10. For rank 3, Sp(3) and SO(7) both have dimension 21. Again, however, the centers are different so they are not isomorphic.

There are some isometries to groups that we did not go over in class. Of particular interest are isometries to the Spin(n) group.

3.3. **Definition.** The real algebra C_k (called the Clifford algebra) of dimension 2^k is generated by $e_1, e_2, \ldots, e_k$ such that $e_i^2 = -1$ and $e_j e_i = -e_i e_j$ if $i \neq j$.

3.4. **Example.** $C_0 = \mathbb{R}$
For C_1, let the basis be $\{1, e\}$.
Let 1 be the multiplicative identity. $e^2 = -1$. Multiplication is
$(a + be)(c + de) = (ac - bd) + (ad + bc)e$
So $C_1 \cong \mathbb{C}$

3.5. **Proposition.** If C_k^* denotes the group of units in C_k, then $S^{k-1} \subset C_k^*$.
(S^{k-1} is the unit sphere in $\mathbb{R}$)

Proof. See [1] p. 135.

□

3.6. **Definition.** Pin(k) is the subgroup of C_k^* generated by S^{k-1}.

3.7. **Definition.** $\alpha(e_i) = -e_i$ is an automorphism of C_k

3.8. **Definition.** For $u \in Pin(k)$ and $x \in \mathbb{R}^k$,
$\rho(u)(x) = \alpha(u)xu^*$
* is conjugation in C_k

3.9. **Proposition.** If $u \in S^{k-1} \subset$Pin(k) and $u \neq \pm 1$,
then $\rho(u)$ is reflection in $\mathbb{R}^k$ in the hyperplane perpendicular to u.

Proof. See [1] p. 136.

□

3.10. **Definition.** Spin(k)= $\rho^{-1}(SO(k))$
From a topological perspective, Spin(n) is the double cover of SO(n).

3.11. **Example.** $C_1 = \mathbb{C}$
$C_1^* = \mathbb{C} - \{0\}$
$S^0 = \{e_1, -e_1\}$
$Pin(1) = \{e_1, e_1^2 = -1, e_1^3 = -e_1, e_1^4 = 1\}$
$\rho(e_1) = \rho(-e_1)$ is the reflection
$Spin(1) = \{1, -1\}$
$Spin(3) = \{a + be_1e_2 + ce_1e_3 + de_2e_3 | a^2 + b^2 + c^2 + d^2 = 1\}$
The assignment
$e_1e_2 \mapsto i$
$e_1e_3 \mapsto j$
$e_2e_3 \mapsto k$
gives an isomorphism Spin(3) $\cong Sp(1)$
Also,
Spin(4) $\cong Sp(1) \times Sp(1)$
Spin(5) $\cong Sp(2)$

Like SO(n), Spin(n) has dimension $\frac{n(n-1)}{2}$. Also, the center of Spin(n) for odd n is $\{\pm 1\}$, so Spin(2n+1) is a good candidate for isomorphism with Sp(n). However, for $n \geq 3$ this is not the case. If Spin(2n+1) $\cong$ Sp(n), then $\frac{Spin(2n+1)}{center} \cong \frac{Sp(n)}{center}$. The normalizer of a maximal torus in $\frac{Sp(n)}{center}$ splits for n=1,2, while the normalizer in a maximal torus for $\frac{Spin(2n+1)}{center}$ splits for all n $\in \mathbb{Z}^+$.

Proof. See chapter 11 of [1]. □

3.12. **Summary.** Here is a summary of isomorphisms of Sp(n) that were mentioned:
Sp(1) $\cong S^3$, SU(2), Spin(3)
Sp(2) $\cong Spin(5)$
SO(3) $\cong Sp(1)/\{\pm I\}$
Spin(4) $\cong Sp(1) \times Sp(1)$
Basically, the isomorphisms occur in low dimensions and are “accidental”.